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Explicit bounds for the positive root of classes of polynomials with applications

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Abstract

We consider a certain type of polynomial equations for which there exists—according to Descartes' rule of signs—only one simple positive root. These equations are occurring in Numerical Analysis when calculating or estimating the R-order or Q-order of convergence of certain iterative processes with an error-recursion of special form. On the other hand, these polynomial equations are very common as defining equations for the effective rate of return for certain cashflows like bonds or annuities in finance. The effective rate of interest i^* for those cashflows is $i^* = q^* - 1$, where q^* is the unique positive root of such polynomial. We construct bounds for i^* for a special problem concerning an ordinary simple annuity which is obtained by changing the conditions of such an annuity with given data applying the German rule (Preisangabeverordnung or short PAngV). Moreover, we consider a number of results for such polynomial roots in Numerical Analysis showing that by a simple variable transformation we can derive several formulas out of earlier results by applying this transformation. The same is possible in finance in order to generalize results to more complicated cashflows.

Keywords: Analytic properties of polynomials; Convergence and divergence of series and sequences; Single equations

1. Introduction

In this paper we are dealing with polynomial equations of the form

$$p(q) = \sum_{t=0}^{n} c_t q^{n-t} = 0 \tag{1}$$

for positive values of $q > 0$. According to Descartes' rule of signs the sign pattern of the real coefficients c_t allow to give an estimate of the number positive roots of Eq. (1).

Let us give some concrete examples for practical occurrences of such equations in two mathematical fields.

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(A) Let us first focus on Numerical Analysis. In the determination of the R-order or Q-order of convergence for a wide class of iterative processes we have to solve the polynomial equation of the type

$$q^n - \sum_{i=1}^{n} a_i q^{n-i} = 0, \quad a_i > 0, \; 1 \leqslant i \leqslant n, \quad \sum_{i=1}^{n} a_i > 1. \tag{2}$$

According to Descartes' rule of signs this equation has exactly one simple positive root q^*. Furthermore, since $q(1) > 0$ holds by assumption, the positive root $q^* > 1$ defines the order of convergence. Results can be found in [11,4] as well as in a collection of other papers, some of them will be cited later on.

(B) Secondly, in classical mathematics of finance we get as defining equation for the effective rate of return of a cashflow, for example for a bond

$$p(x) = Cx^n - \sum_{j=1}^{n-1} B_j x^{n-j} - A = 0, \quad x \geqslant 0 \quad \text{with } A, B_j, C > 0. \tag{3}$$

Let x^* be the unique positive root (according to Descartes' rule), then we can calculate the effective rate of interest i^* through $i^* = 1/x^* - 1$.
In Eq. (3) A means the purchase price of the bond, B_j, $1 \leqslant j \leqslant n-1$, (usually equal a constant amount of B) are the periodic payments according to the contract rate and $C = B + K$, where K is the purchase rate of the bond when sold at the bond market or the redemption value when surrendered to the issuer. The integer number n is the term of the bond measured usually in full years. Results for estimating i^* can be found for example in [5]. In this paper we do not treat Eq. (3).

Another example for polynomial equations in finance are those derived for the calculation of the effective rate of annuities. For an ordinary simple annuity which is a very common form of a consumer loan we get as equation when applying the US-rule

$$Sq^n - A \sum_{j=1}^{n} q^{n-j} = 0$$

or in a different way of writing

$$q^n - a \sum_{j=1}^{n} q^{n-j} = 0, \quad a = \frac{A}{S}. \tag{4}$$

At a first glance Eq. (4) is showing a striking resemblance to Eq. (1). In Eq. (4) the quantity S means the present value of the annuity or simply speaking the amount of the loan, A is the amount of the periodic payments, i the interest rate of the annuity, q is defined as $q = i + 1$ and lastly, n is the number of periodic payments which is equal to the term of the annuity according to the US-rule. At the end of the term of the annuity the loan is usually paid back together with all interests due. It should be mentioned that for an annuity usually we have $i^* > 0$ and thus the inequality $n \cdot A > S$ or equivalently $n \cdot a > 1$ is fulfilled. This is in accordance with the assumption made by [13] when deriving bounds for the similar Eq. (1) in Numerical Analysis.

At this stage we want to mention one result of Petković and Petković [10] which will draw our attention at several considerations later on. This interesting result proven by means of Linear Algebra, especially by extensions of the Perron–Frobenius Theory, uses one result of E. Deutsch and gives the lower bound for the positive root of Eq. (4) in the form of

$$q^* > \frac{(2a + a(n-1)(n+2))}{(2 + a(n-1)n)}.$$

The formula above is specialized to our Eq. (4) since it was originally derived for a more general equation in the theory of the calculation of the order of convergence of some iterative numerical processes. The equations considered there are containing our Eq. (4) as a special case setting one parameter equal to one. The main advantage of the formula of Petković and Petković [10] was, besides its comparatively simple structure, the absence of the assumption $n \cdot a > 1$, which is always true in the application to iterative processes, but can sometimes not be fulfilled when considering for example annuities with geometrically growing payments for certain growth factors. In those cases the classical results, usually derived by applying methods from analysis, cannot be extended without substantial new proofs.

2. Treating a special problem for annuities

Given an ordinary simple annuity with the present value of S and annual payments of equal amounts A during the term of n years. Its interest rate should be denoted by i (in decimals). In practice, the following question arose when considering a slightly changed annuity ([9], Federal Ministry of Finance):

> *The changed annuity should have regular payments of amount* A/k, $k > 1$, *but the number of payments should be increased during the same term by a factor of* k (*by reducing the length each payment interval by the factor of* $1/k$). *What can be said about the effective rate of return* j^* *of the in this way changed annuity in comparison to the known interest rate* i^* *of the original annuity*?

The problem was treated and solved by using the US-rule in [6]. More difficult to treat is when applying the German rule or Preisangabenverordnung (PAngV) since the changed annuity has several payments during a year and thus its defining equation for the effective rate of interest j^* is not of the same structure (and degree) as the corresponding equation for the US-rule. For annual payments in both cases we get the same equation. But in our case the original annuity with annual payments corresponds to an equation according to the US-rule of higher degree, whereas the changed annuity corresponds to an equation of different structure (and lower degree) since with its subannual payment intervals the derivation of the corresponding equation does not match with that according to the US-rule.

Since the compounding time intervals for the PAngV are always full years, the payments during a year are not balanced like in the US-rule but used to lower the debt. The interest due of the payments for the rest of the year is calculated by applying the linear proportional interest rate. This means that we just have to calculate a fictive yearly payment as the sum of the k payments during the year including their interest due. This can then plugged into the usual formula for annuities with annual payments.

The fictive annual payment is given by

$$\bar{A} = \sum_{v=0}^{k-1} \frac{A}{k}(1 + i_k v) = \frac{A}{2k}((k+1) + (k-1)q),$$

where $i_k = i/k$ is the linear proportional interest rate.

This leads to the new polynomial equation for the changed annuity according to the PAngV

$$p_k(q) = q^n - \bar{a} \sum_{v=1}^{n-1} q^{n-v} - \bar{b} = 0 \tag{5}$$

with

$$\bar{a} = \frac{2Ak}{2kS - A(k-1)} \quad \text{and} \quad \bar{b} = \frac{A(k+1)}{2kS - A(k-1)}.$$

Our original question is now posed as the mathematical question:

What is the relation between the unique positive root of Eq. (4) *which determines the interest rate known to be i and the unique positive root of Eq.* (5) *which determines the unknown effective rate j^* of the changed annuity*?

The intention of our consideration is to establish a kind of comparatively simple to calculate set of formulas for a pair of upper and lower bounds for j^* in dependence of the parameters A, S, n, k and the known interest rate i of the original annuity. In order to get an idea what the dependence could be, we consider the limit of j^* dependent on n for $n \to \infty$. This turns out to be

$$\lim_{n\to\infty} j^* = \frac{2ki}{2k - i(k-1)}.$$

Some numerical examples with a set of reasonable data for the parameters in Eq. (5) are showing that in these particular cases always

$$j^* > \frac{2ki}{2k - i(k-1)}$$

holds true. Posed as a conjecture, the last inequality can then be proved by simply determining the sign of the polynomial value of p_k at the abscissa $1 + 2ki/(2k - i(k-1))$. This value can be assured to be always negative which means, that the unique positive root of Eq. (5) is located on the right-hand side of this abscissa, proving the above conjectured inequality. Again by estimating the polynomial value of p_k in (5) at the abscissa $1 + 2Ak/(2kS - A(k-1))$, we can again prove by some elementary calculations that its sign is always positive. In terms of $j^* = q^* - 1$, we thus get the initial bounds for j^* as

$$\frac{2ki}{2k - i(k-1)} < j^* < \frac{2kA}{2kS - A(k-1)}.$$

We now observe that the polynomial p_k in (5) is convex in the region under consideration. Thus, we are able to improve these first bounds by taking one explicit step of the secant method with the two bounds given above as starting points. This gives then necessarily an improved lower bound for j^* by the following formula:

$$j^* > \frac{2ki}{2k-i(k-1)} - \frac{\frac{2ki}{2k-i(k-1)} - \frac{2Ak}{2kS-A(k-1)}}{1 - \frac{iS}{\left[(1+\frac{2ki}{2k-i(k-1)})^n(iS-A)\right]+A}}.$$

In the same way we can calculate one explicit step of Newton's method starting with the expression for the first given upper bound for j^* and following the same argumentation we thus get an improved upper bound for j^* by

$$j^* < t - \frac{\frac{(1+t)^n 2(ikS-kA)+2Ak}{i(2kS-A(k-1))}}{(1+t)^{n-1}\left[n - \frac{2Ak}{2kS-A(k-1)} \cdot \frac{(n-1)(1+t)-n}{t^2}\right] - \frac{\frac{2Ak}{2kS-A(k-1)}}{t^2}},$$

where

$$t = \frac{2k}{2k-i(k-1)}.$$

Let us demonstrate the accuracy of these new bounds just by giving a numerical example.

Example 1. $n=5,\ k=2,\ i=0.0375.$

For these data we get the interval

$$0.039853937512 < j^* < 0.0411382583.$$

These bounds can be compared with the "exact" value of j^* approximated by Newton's method in floating-point arithmetic

$$j^* = 0.0411382583.$$

The given results can be found partly in [2].

3. Transformation of bounds for positive roots

We first give a collection of bounds for the positive root of polynomials out of the literature.

The first important result about bounds for the polynomial equation like in (4) were given in [13] in connection with the determination of the Q-order of convergence of special iteration methods for root-finding, the so-called interpolatory iteration methods. Special representatives are the well-known secant method, the Newton and Müller's methods. For $n \cdot a > 1$ it is proved that

$$a+1-\frac{a}{(a+1)^n}\left(1+\frac{1}{n}\right)^n < \tau < a+1-\frac{a}{(a+1)^n}$$

holds true (see [13, Lemmas 3–9]) where τ is the positive root of Eq. (4).

The proof of this result is done by simple analysis and the trick to consider the polynomial of one more degree

$$(q-1)\left(q^n - a\sum_{j=1}^{n} q^{n-j}\right) = q^{n+1} - (a+1)q^n + a,$$

the roots of which are those of the polynomial in (4) including a second positive root 1. The discussion of the latter polynomial turns out to be technically simpler. Much more later, Kjurkchiev [8] was treating polynomials of the form

$$q^n - (p+1)\sum_{i=0}^{n-1} h^i q^{n-i-1} = 0, \tag{6}$$

which arose in calculating the order of convergence of some special classes of iteration methods (see [4]). The result are the following bounds:

$$p+h+1-\frac{(p+1)h^n}{(p+h+1)^n}\left(1+\frac{1}{n}\right)^n < \tau < p+h+1-\frac{(p+1)h^n}{(p+h+1)^n}$$

$$\text{for all } n > \frac{h}{(p+1)}.$$

The structure of both polynomials in (4) and (6) as well as the structure of the corresponding bounds for its positive root τ given in [13] or [8] show some remarkable similarity.

This can be used to try to convert the polynomial in (4) into that in (6) by applying a simple variable transformation [7]. If this works, then we could try to get the bounds given by Kjurkchiev out of those by Traub by means of the same transformation.

First, let us study a transformation with polynomials like in (6). We denote the polynomial in (6) by

$$p(q) = q^n - (p+1)\sum_{i=0}^{n-1} h^i q^{n-i-1}.$$

Then we divide this polynomial by $h^n > 0$ and get

$$\frac{p(q)}{h^n} = \left(\frac{q}{h}\right)^n - h(p+1)\sum_{i=0}^{n-1}\left(\frac{q}{h}\right)^{n-i-1}.$$

If we now set the new variable $x = q/h$, then we get the new polynomial p/h^n as

$$r(x) = x^n - \tilde{a}\sum_{i=0}^{n-1} x^{n-i} \quad \text{where } \tilde{a} = h(p+1).$$

This, however, is exactly the form of the polynomial in (4).

Now, we can take the bounds given in [13] for the positive root q^* of the polynomial in (4) and try to transform it by setting

$$\tilde{a} = h(p+1) \quad \text{and} \quad x = \frac{q^*}{h}$$

in order to derive bounds for the positive root of the transformed polynomial which is just the polynomial in (6)

$$h(p+1)+1-\frac{h(p+1)}{(h(p+1)+1)^n}\left(1+\frac{1}{n}\right)^n < \frac{q^*}{h} < h(p+1)+1-\frac{h(p+1)}{(h(p+1)+1)^n}.$$

A simplification of the expressions for this bounds then leads exactly to the bounds given in [8].

It should be mentioned here that one carefully has to check that using this transformation applied to the polynomial in (4) does not affect the conclusions in the analysis of the proof given in [13]. It can be shown that this is the case here.

This last example shows how bounds for the positive root of polynomials given in the literature can be changed into each other by a simple variable transformation. The next case gives an example where such a variable transformation can be used to simplify the proof of a result in the literature by reducing the consideration to a more simpler polynomial.

In [3] the polynomial equation is considered

$$p(q) = q^n - aq^{n-1} - b\sum_{i=1}^{n-1} s^i q^{n-1-i} = 0, \quad a>0,\ b>0,\ s>0. \tag{7}$$

For the unique positive root of this kind of polynomials we find the bounds valid for $n=n_0(a,b,s) \geqslant 2$

$$\tfrac{1}{2}(n+1)^{-1}[n(a+s)+\sqrt{g^2n^2+4s(a-b)}] < q^* < \tfrac{1}{2}(a+s+|g|),$$

where $g^2 = (a+s)^2 - 4s(a-b)$.

Abels [1] showed that in a similar way these bounds can easily be derived by simply making the whole analysis necessary for the proof just for the polynomial with $s=1$ which is the transformed polynomial of p in (7). Therefore, we can consider the polynomial r

$$r(x) = x^n - \tilde{a}x^{n-1} - \tilde{b}\sum_{i=1}^{n-1} x^{n-1-i} \quad \text{with} \quad \tilde{a} = \frac{a}{s},\ \tilde{b} = \frac{b}{s}.$$

We want to sketch the proof for the bounds for the positive root of polynomial r in (7) following the line given in the paper Herzberger [3].

First we consider the polynomial

$$\hat{r}(x) = (x-1)r(x) = x^n - (\tilde{a}+1)x^{n-1} + (\tilde{a}-\tilde{b})x + \tilde{b}.$$

It can be shown that this polynomial has the following properties:

(i) $\hat{r}$ has the same roots like r and additionally the root 1.
(ii) $\hat{r}$ has for $n \geqslant 2g^{-1}\sqrt{|\tilde{a}-\tilde{b}|}$ and $\tilde{a} \neq \tilde{b}$ the only roots different from 0

$$x_{1,2} = \tfrac{1}{2}(n+1)^{-1}\left[n(\tilde{a}+1) \pm \sqrt{g^2n^2+4(\tilde{a}-\tilde{b})}\right] \quad \text{where } |g| = \sqrt{(\tilde{a}+1)^2 - 4(\tilde{a}-\tilde{b})}.$$

(iii) $r'(1) < 0$ iff $n > ((1-\tilde{a})/\tilde{b}) + 1$.
(iv) q^* is strictly monotonic increasing with growing n.

By applying the analysis similar to that made in [3] we then can prove the final inequalities:

(a) For $n \geqslant n_0(a, b)$ we have

$$\tfrac{1}{2}(n+1)^{-1}\left[n(a+1) \pm \sqrt{n^2 g^2 + 4(\tilde{a} - \tilde{b})}\right] < q^*.$$

(b) Then it follows from (iv) that, on the other hand, we have

$$\lim_{n\to\infty} q^*(n) = \tfrac{1}{2}(a+1+|g|) > q^*.$$

Now, we just have to apply the transformation $x = q^*/s$, $\tilde{a} = a/s$ and $\tilde{b} = b/s$ and after a simplification of expressions we get the original bounds in [3].

The next example considers a result originally derived in [12], but the analysis and proof require quite complicated conclusions. Siebenbrodt's result is concerned with the defining equation of an annuity with geometrically growing payments according to the formula

$$A_j = h^{j-1}A, \quad 1, 2, \ldots, n.$$

For this case we get a generalization of Eq. (4) from Section 2

$$p(q) = q^n - a\sum_{v=1}^{n} h^{v-1}q^{n-v} = 0, \quad a > 0,\ h > 0,\ n \geqslant 2. \tag{8}$$

Here we again set $a = A/S$.

Bounds for this kind of equations were first given in [8] in connection with the determination of the order of convergence of iterative numerical processes. Like in the original proof of the result of Traub, we can always assume that $n > s/a$ in those equations. Therefore, the results of Kjurkchiev as well as those of Traub are only valid in this case and their analysis is strongly dependent on this assumption. But in the treatment of annuities in mathematics of finance one often gets the case $n \leqslant h/a$ depending on the size of the growth factor $h > 0$.

In order to get proper bounds for the unique positive root q^* of Eq. (8) Siebenbrodt makes a more sophisticated analysis and distinguishes between two possible cases. It can easily be shown that the variable transformation $x = q/h$ does not afflict this situation. Thus, we can restrict the proof to a more simpler case with $h = 1$ of Eq. (8) and then transform the resulting bounds into those for the general equation (8). This means that we have to deal with one parameter less in Eq. (8) by setting $h = 1$. The transformed equation reads as

$$\tilde{p}(q) = q^n - \tilde{a}\sum_{j=0}^{n-1} q^{n-j-1}, \quad \tilde{a} = \frac{a}{h},\ n \geqslant 2. \tag{9}$$

One can show that in both cases considered in [12], namely for the cases

$$-2(\tilde{a} + h)^{n+1}\frac{(n-1)^n}{(n+1)^{n+1}} + \tilde{a}h^n > 0 \quad \text{or} \quad \leqslant 0,$$

these bounds for (9) are exactly transformed into the resulting bounds given in [12].

Case 1: $-2(\tilde{a}+1)^{n+1}[(n-1)^n/(n+1)^{n+1}]+\tilde{a}h^n>0$:

$$q_1=\tilde{a}\left(\frac{1}{\tilde{a}+1}\right)^n\left(\frac{n-1}{n-1}\right)^{n-1}-\frac{2(n-1)(\tilde{a}+1)}{(n+2)^2}+\frac{n-1}{n+1}(\tilde{a}+1),$$

$$q_2=\frac{n(\tilde{a}+1)}{n+1}+\frac{\tilde{a}(\frac{n+1}{\tilde{a}+1})^n-\frac{\tilde{a}+1}{n+1}n^n}{n^n-2(n-1)^n}. \tag{10}$$

Case 2: $-(\tilde{a}+1)^{n+1}[(n-1)^n/(n+1)^{n+1}]+\tilde{a}h^n\leqslant 0$:

$$q_1=\frac{\tilde{a}}{2}\left(\frac{1}{\tilde{a}+1}\right)^n\frac{(n+1)^n}{(n-1)^{n-1}},$$

$$q_2=\tilde{a}\left(\frac{1}{\tilde{a}+1}\right)^n\left(\frac{n+1}{n-1}\right)^{n-1}-\frac{2(n-1)(\tilde{a}+1)}{(n+1)^2}+\frac{n-1}{n+1}(\tilde{a}+1).$$

Now, we want to sketch the technical proof for the bounds (10), i.e. basically for the case $h=1$ of Eq. (8).

The case $na>1$ has already been treated in [13]. Therefore, it remains to focus on the more complicated case $na\leqslant 1$.

Instead of the original polynomial p we consider the polynomial

$$r(q)=(q-1)\tilde{p}(q)=q^{n+1}-(\tilde{a}+1)q^n+\tilde{a},$$

which is of one degree higher than $\tilde{p}$. The roots of r exactly are those of $\tilde{p}$ with the additional positive root $q=1$. In our case we always have $r'(1)\geqslant 0$, since the derivative r' is

$$r'(q)=q^{n-1}((n+1)q-n(\tilde{a}+1)).$$

This means that the new positive root introduced by enlarging the degree of the polynomial p is always located on the right-hand side of the positive root q^* under consideration.

Obviously, the only nonzero (and positive) root or the derivative r' of the polynomial r is

$$q_1=\frac{n}{n+1}(a+1).$$

Fig. 1 should roughly visualize the shape of the graph of the polynomial r on the positive q-axis.

The second derivative r'' of the polynomial r is

$$r''(q)=nq^{n-2}((n+1)q-(n-1)(\tilde{a}+1))$$

and its only nonzero (and positive) root q_2 is

$$q_2=\frac{n-1}{n+1}(\tilde{a}+1)=q_1-\frac{\tilde{a}+1}{n+1}.$$

Since $q^*<\tilde{a}+1$ obviously holds true and on the other hand $r''(\tilde{a}+1)=2n(\tilde{a}+1)^{n-1}>0$, the polynomial r is convex in the interval (q_2,∞) and concave in the interval $(0,q_2)$.

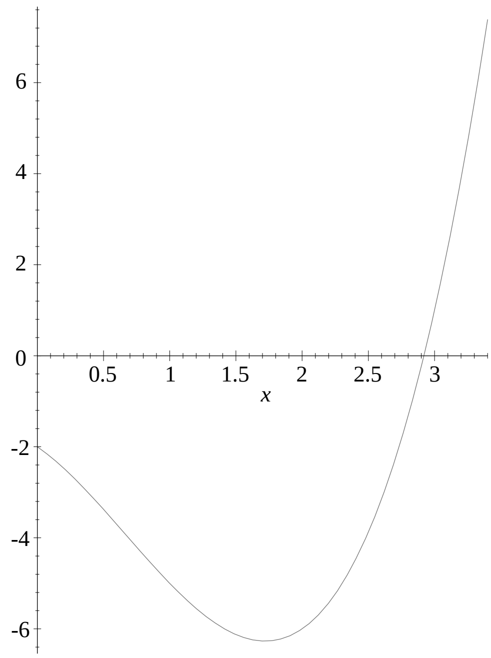

Fig. 1. Graph of $p(q) = q^3 - 2q^2 - 2q - 2$.

Since $q_1 > q^*$, the value of q_1 already is a lower bound for the positive zero q^*. But this bound still has to be improved.

Now, we have to treat two possible cases for the location of q_2 with respect to the positive root q^*. These are the cases $q_2 \geqslant q^*$ (or equivalently $r(q_2) \leqslant 0$) and $q_2 < q^*$ (or equivalently $r(q_2) > 0$). Then we proceed as follows in order to improve the bounds for q^*. For example in the first case, the straight line between the points $(0, r(0))$ and $(q_2, r(q_2))$ cuts the abscissa in the new (and improved) lower bound y_1 for q^*. On the other hand, the cutting point between the tangent on the graph of r at point $(q_2, r(q_2))$ gives an upper bound for q^*. Similar procedures are done in the case $q_2 < q^*$, where now y_1 is the cutting point of the tangent with the abscissa and y_2 the cutting point of the straight line between the points $(q_1, r(q_1))$ and $(q_2, r(q_2))$ with the abscissa.

After elementary but lengthy calculations one gets the bounds as stated in (10). A simple numerical example should demonstrate the quality of the bounds in (10).

Example 2. We choose $n = 4$ and $a = 0.2$. This leads to the polynomial equation

$$r(q) = q^5 - 1.2q^4 + 0.2.$$

The application of MAPLE calculates approximately $q^* = 0.9163545825$. For q^* we get the following bounds:

$$0.8785306351 < q^* < 0.9476769779.$$

(This is case 1 for the bounds (10).)

Example 3. Next, we choose $n = 3$ and $a = 0.03$ and get the polynomial

$$r(q) = q^4 - 1.03q^3 + 0.03.$$

Again with MAPLE we get the approximation $q^* = 0.3540542425$ and the bounds are

$$0.2196339982 < q^* < 0.3673169994.$$

(This is case 2 for the bounds (10).)

4. Conclusions

The result in Section 2 concerning the changed annuity with subannual payments according to the PAngV is published first in this paper. The same is true for the transformed bounds according to Herzberger [3] and the proof for the transformed bounds of Siebenbrodt's result, both in Section 3.

The transformation used here to convert simpler cases into more complicated ones with additional growth parameter h does not work out in all cases of derived bounds in the literature. Let us, for example, consider the bounds given in [10] for the positive root of Eq. (7). It is given by the formula

$$\frac{a + b(S_1 + S_2)}{1 + bS_1} < q^*,$$

where

$$S_1 = \begin{cases} \frac{n(n-1)}{2} & \text{for } s = 1, \\ \frac{s[(n-1)s^n - ns^{n-1} + 1]}{(s-1)^2} & \text{for } s \neq 1, \end{cases} \qquad S_2 = \begin{cases} n - 1 & \text{for } s = 1, \\ \frac{s^n - s}{s-1} & \text{for } s \neq 1. \end{cases}$$

The techniques used for the proof of this bound makes strong use of Linear Algebra methods as already mentioned in Section 1. Up to now no successful application of such a variable transformation was found being compatible with techniques used in the proof regarding methods from Linear Algebra. In other words, the derivation of the formulas of Petković and Petković [10] seems to be resistent against such a variable transformation approach. Nevertheless, the already existing cases, where this technique worked out quite successfully, are encouraging and bringing enough progress to justify this method. A few of these successful applications were described in this paper.

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